9.2 Arithmetic Sequences and Partial Sums

Arithmetic sequences have many real-life applications. For example, in Exercise 73 on page 627, you will use an arithmetic sequence to determine how far an object falls in 7 seconds when dropped from the top of the Willis Tower in Chicago.

- **Recognize, write, and find the *n*th terms of arithmetic sequences.**
- **Find *n*th partial sums of arithmetic sequences.**
- **Use arithmetic sequences to model and solve real-life problems.**

Arithmetic Sequences

A sequence whose consecutive terms have a common difference is an **arithmetic sequence.**

Definition of Arithmetic Sequence

A sequence is **arithmetic** when the differences between consecutive terms are the same. So, the sequence

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

is arithmetic when there is a number d such that

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots = d.$$

The number d is the **common difference** of the arithmetic sequence.

EXAMPLE 1 Examples of Arithmetic Sequences

a. The sequence whose nth term is $4n + 3$ is arithmetic. The common difference between consecutive terms is 4.

$$\underbrace{7, 11}_{11-7=4}, 15, 19, \ldots, 4n + 3, \ldots$$ Begin with $n = 1$.

b. The sequence whose nth term is $7 - 5n$ is arithmetic. The common difference between consecutive terms is -5.

$$\underbrace{2, -3}_{-3-2=-5}, -8, -13, \ldots, 7 - 5n, \ldots$$ Begin with $n = 1$.

c. The sequence whose nth term is $\frac{1}{4}(n + 3)$ is arithmetic. The common difference between consecutive terms is $\frac{1}{4}$.

$$\underbrace{1, \frac{5}{4}}_{\frac{5}{4}-1=\frac{1}{4}}, \frac{3}{2}, \frac{7}{4}, \ldots, \frac{n+3}{4}, \ldots$$ Begin with $n = 1$.

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Write the first four terms of the arithmetic sequence whose nth term is $3n - 1$. Then find the common difference between consecutive terms.

The sequence 1, 4, 9, 16, . . . , whose nth term is n^2, is *not* arithmetic. The difference between the first two terms is

$$a_2 - a_1 = 4 - 1 = 3$$

but the difference between the second and third terms is

$$a_3 - a_2 = 9 - 4 = 5.$$

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The nth term of an arithmetic sequence can be derived from the pattern below.

$a_1 = a_1$ — 1st term

$a_2 = a_1 + d$ — 2nd term

$a_3 = a_1 + 2d$ — 3rd term

$a_4 = a_1 + 3d$ — 4th term

$a_5 = a_1 + 4d$ — 5th term

1 less

⋮

$a_n = a_1 + (n - 1)d$ — nth term

1 less

The following definition summarizes this result.

The nth Term of an Arithmetic Sequence

The nth term of an arithmetic sequence has the form

$$a_n = a_1 + (n - 1)d$$

where d is the common difference between consecutive terms of the sequence and a_1 is the first term.

EXAMPLE 2 Finding the nth Term

Find a formula for the nth term of the arithmetic sequence whose common difference is 3 and whose first term is 2.

Solution You know that the formula for the nth term is of the form $a_n = a_1 + (n - 1)d$. Moreover, the common difference is $d = 3$ and the first term is $a_1 = 2$, so the formula must have the form

$$a_n = 2 + 3(n - 1). \quad \text{Substitute 2 for } a_1 \text{ and 3 for } d.$$

So, the formula for the nth term is $a_n = 3n - 1$.

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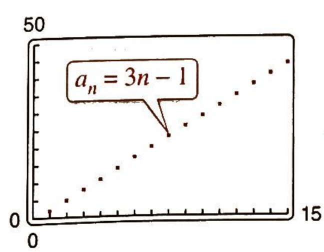
Find a formula for the nth term of the arithmetic sequence whose common difference is 5 and whose first term is -1.

The sequence in Example 2 is as follows.

$$2, 5, 8, 11, 14, \ldots, 3n - 1, \ldots$$

The figure below shows a graph of the first 15 terms of this sequence. Notice that the points lie on a line. This makes sense because a_n is a linear function of n. In other words, the terms "arithmetic" and "linear" are closely connected.



REMARK Another way to find a_1 in Example 3 is to use the definition of the nth term of an arithmetic sequence, as shown below.

$$a_n = a_1 + (n - 1)d$$
$$a_4 = a_1 + (4 - 1)d$$
$$20 = a_1 + (4 - 1)5$$
$$20 = a_1 + 15$$
$$5 = a_1$$

EXAMPLE 3 Writing the Terms of an Arithmetic Sequence

The 4th term of an arithmetic sequence is 20, and the 13th term is 65. Write the first 11 terms of this sequence.

Solution You know that $a_4 = 20$ and $a_{13} = 65$. So, you must add the common difference d nine times to the 4th term to obtain the 13th term. Therefore, the 4th and 13th terms of the sequence are related by

$$a_{13} = a_4 + 9d. \qquad a_4 \text{ and } a_{13} \text{ are nine terms apart.}$$

Using $a_4 = 20$ and $a_{13} = 65$, you have $65 = 20 + 9d$. Solve for d to find that the common difference is $d = 5$. Use the common difference with the known term a_4 to write the other terms of the sequence.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11} . . .
5	10	15	20	25	30	35	40	45	50	55 . . .

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The 8th term of an arithmetic sequence is 25, and the 12th term is 41. Write the first 11 terms of this sequence.

When you know the nth term of an arithmetic sequence *and* you know the common difference of the sequence, you can find the $(n + 1)$th term by using the *recursion formula*

$$a_{n+1} = a_n + d. \qquad \text{Recursion formula}$$

With this formula, you can find any term of an arithmetic sequence, *provided* that you know the preceding term. For example, when you know the first term, you can find the second term. Then, knowing the second term, you can find the third term, and so on.

EXAMPLE 4 Using a Recursion Formula

Find the ninth term of the arithmetic sequence whose first two terms are 2 and 9.

Solution The common difference between consecutive terms of this sequence is

$$d = 9 - 2 = 7.$$

There are two ways to find the ninth term. One way is to write the first nine terms (by repeatedly adding 7).

2, 9, 16, 23, 30, 37, 44, 51, 58

Another way to find the ninth term is to first find a formula for the nth term. The common difference is $d = 7$ and the first term is $a_1 = 2$, so the formula must have the form

$$a_n = 2 + 7(n - 1). \qquad \text{Substitute 2 for } a_1 \text{ and 7 for } d.$$

Therefore, a formula for the nth term is

$$a_n = 7n - 5$$

which implies that the ninth term is

$$a_9 = 7(9) - 5$$
$$= 58.$$

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Find the 10th term of the arithmetic sequence that begins with 7 and 15.

The Sum of a Finite Arithmetic Sequence

There is a formula for the *sum* of a finite arithmetic sequence.

REMARK Note that this formula works only for *arithmetic* sequences.

The Sum of a Finite Arithmetic Sequence

The sum of a finite arithmetic sequence with n terms is given by $S_n = \frac{n}{2}(a_1 + a_n)$.

For a proof of this formula, see Proofs in Mathematics on page 687.

EXAMPLE 5 **Sum of a Finite Arithmetic Sequence**

Find the sum: $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$.

Solution To begin, notice that the sequence is arithmetic (with a common difference of 2). Moreover, the sequence has 10 terms. So, the sum of the sequence is

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad \text{Sum of a finite arithmetic sequence}$$

$$= \frac{10}{2}(1 + 19) \qquad \text{Substitute 10 for } n\text{, 1 for } a_1\text{, and 19 for } a_n.$$

$$= 5(20) = 100. \qquad \text{Simplify.}$$

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Find the sum: $40 + 37 + 34 + 31 + 28 + 25 + 22$.

EXAMPLE 6 **Sum of a Finite Arithmetic Sequence**

Find the sum of the integers (a) from 1 to 100 and (b) from 1 to N.

Solution

a. The integers from 1 to 100 form an arithmetic sequence that has 100 terms. So, use the formula for the sum of a finite arithmetic sequence.

$$S_n = 1 + 2 + 3 + 4 + 5 + 6 + \cdots + 99 + 100$$

$$= \frac{n}{2}(a_1 + a_n) \qquad \text{Sum of a finite arithmetic sequence}$$

$$= \frac{100}{2}(1 + 100) \qquad \text{Substitute 100 for } n\text{, 1 for } a_1\text{, and 100 for } a_n.$$

$$= 50(101) = 5050 \qquad \text{Simplify.}$$

b. $S_n = 1 + 2 + 3 + 4 + \cdots + N$

$$= \frac{n}{2}(a_1 + a_n) \qquad \text{Sum of a finite arithmetic sequence}$$

$$= \frac{N}{2}(1 + N) \qquad \text{Substitute } N \text{ for } n\text{, 1 for } a_1\text{, and } N \text{ for } a_n.$$

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Find the sum of the integers (a) from 1 to 35 and (b) from 1 to $2N$.

A teacher of Carl Friedrich Gauss (1777–1855) asked him to add all the integers from 1 to 100. When Gauss returned with the correct answer after only a few moments, the teacher could only look at him in astounded silence. This is what Gauss did:

$$S_n = 1 + 2 + 3 + \cdots + 100$$
$$S_n = 100 + 99 + 98 + \cdots + 1$$
$$2S_n = 101 + 101 + 101 + \cdots + 101$$
$$S_n = \frac{100 \times 101}{2} = 5050.$$

CORBIS

Recall that the sum of the first n terms of an infinite sequence is the *nth partial sum*. The nth partial sum of an arithmetic sequence can be found by using the formula for the sum of a finite arithmetic sequence.

EXAMPLE 7 **Partial Sum of an Arithmetic Sequence**

Find the 150th partial sum of the arithmetic sequence

5, 16, 27, 38, 49, ...

Solution For this arithmetic sequence, $a_1 = 5$ and $d = 16 - 5 = 11$. So,

$$a_n = 5 + 11(n - 1)$$

and the nth term is

$$a_n = 11n - 6.$$

Therefore, $a_{150} = 11(150) - 6 = 1644$, and the sum of the first 150 terms is

$$S_{150} = \frac{n}{2}(a_1 + a_{150}) \qquad n\text{th partial sum formula}$$

$$= \frac{150}{2}(5 + 1644) \qquad \text{Substitute 150 for } n, 5 \text{ for } a_1, \text{ and 1644 for } a_{150}.$$

$$= 75(1649) \qquad \text{Simplify.}$$

$$= 123{,}675. \qquad n\text{th partial sum}$$

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Find the 120th partial sum of the arithmetic sequence

6, 12, 18, 24, 30, ...

EXAMPLE 8 **Partial Sum of an Arithmetic Sequence**

Find the 16th partial sum of the arithmetic sequence

100, 95, 90, 85, 80, ...

Solution For this arithmetic sequence, $a_1 = 100$ and $d = 95 - 100 = -5$. So,

$$a_n = 100 + (-5)(n - 1)$$

and the nth term is

$$a_n = -5n + 105.$$

Therefore, $a_{16} = -5(16) + 105 = 25$, and the sum of the first 16 terms is

$$S_{16} = \frac{n}{2}(a_1 + a_{16}) \qquad n\text{th partial sum formula}$$

$$= \frac{16}{2}(100 + 25) \qquad \text{Substitute 16 for } n, 100 \text{ for } a_1, \text{ and 25 for } a_{16}.$$

$$= 8(125) \qquad \text{Simplify.}$$

$$= 1000. \qquad n\text{th partial sum}$$

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Find the 30th partial sum of the arithmetic sequence

78, 76, 74, 72, 70, ...

Application

EXAMPLE 9 **Total Sales**

See LarsonPrecalculus.com for an interactive version of this type of example.

A small business sells \$10,000 worth of skin care products during its first year. The owner of the business has set a goal of increasing annual sales by \$7500 each year for 9 years. Assuming that this goal is met, find the total sales during the first 10 years this business is in operation.

Solution When the goal is met the annual sales form an arithmetic sequence with

$$a_1 = 10{,}000 \quad \text{and} \quad d = 7500.$$

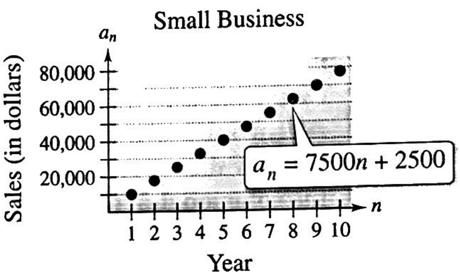
So,

$$a_n = 10{,}000 + 7500(n - 1)$$

and the nth term of the sequence is

$$a_n = 7500n + 2500.$$

Therefore, the 10th term of the sequence is



$$a_{10} = 7500(10) + 2500$$
$$= 77{,}500. \qquad \text{See figure.}$$

The sum of the first 10 terms of the sequence is

$$S_{10} = \frac{n}{2}(a_1 + a_{10}) \qquad n\text{th partial sum formula}$$

$$= \frac{10}{2}(10{,}000 + 77{,}500) \qquad \text{Substitute 10 for } n, \text{ 10,000 for } a_1, \text{ and 77,500 for } a_{10}.$$

$$= 5(87{,}500) \qquad \text{Simplify.}$$

$$= 437{,}500. \qquad \text{Multiply.}$$

So, the total sales for the first 10 years will be \$437,500.

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A company sells \$160,000 worth of printing paper during its first year. The sales manager has set a goal of increasing annual sales of printing paper by \$20,000 each year for 9 years. Assuming that this goal is met, find the total sales of printing paper during the first 10 years this company is in operation.

Summarize (Section 9.2)

1. State the definition of an arithmetic sequence *(page 620)*, and state the formula for the nth term of an arithmetic sequence *(page 621)*. For examples of recognizing, writing, and finding the nth terms of arithmetic sequences, see Examples 1–4.
2. State the formula for the sum of a finite arithmetic sequence and explain how to use it to find the nth partial sum of an arithmetic sequence *(pages 623 and 624)*. For examples of finding sums of arithmetic sequences, see Examples 5–8.
3. Describe an example of how to use an arithmetic sequence to model and solve a real-life problem *(page 625, Example 9)*.