# 9.3 Geometric Sequences and Series

Geometric sequences can help you model and solve real-life problems. For example, in Exercise 84 on page 636, you will use a geometric sequence to model the population of Argentina from 2009 through 2015.

- Recognize, write, and find the $n$th terms of geometric sequences.
- Find the sum of a finite geometric sequence.
- Find the sum of an infinite geometric series.
- Use geometric sequences to model and solve real-life problems.

## Geometric Sequences

In Section 9.2, you learned that a sequence whose consecutive terms have a common *difference* is an arithmetic sequence. In this section, you will study another important type of sequence called a **geometric sequence.** Consecutive terms of a geometric sequence have a common *ratio.*

**REMARK** Be sure you understand that a sequence such as 1, 4, 9, 16, . . . , whose $n$th term is $n^2$, is *not* geometric. The ratio of the second term to the first term is

$$\frac{a_2}{a_1} = \frac{4}{1} = 4$$

but the ratio of the third term to the second term is

$$\frac{a_3}{a_2} = \frac{9}{4}.$$

### Definition of Geometric Sequence

A sequence is **geometric** when the ratios of consecutive terms are the same. So, the sequence $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ is geometric when there is a number $r$ such that

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \cdots = r, \quad r \neq 0.$$

The number $r$ is the **common ratio** of the geometric sequence.

### EXAMPLE 1 Examples of Geometric Sequences

**a.** The sequence whose $n$th term is $2^n$ is geometric. The common ratio of consecutive terms is 2.

$$2, 4, 8, 16, \ldots, 2^n, \ldots \qquad \frac{4}{2} = 2$$

Begin with $n = 1$.

**b.** The sequence whose $n$th term is $4(3^n)$ is geometric. The common ratio of consecutive terms is 3.

$$12, 36, 108, 324, \ldots, 4(3^n), \ldots \qquad \frac{36}{12} = 3$$

Begin with $n = 1$.

**c.** The sequence whose $n$th term is $\left(-\frac{1}{3}\right)^n$ is geometric. The common ratio of consecutive terms is $-\frac{1}{3}$.

$$-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots, \left(-\frac{1}{3}\right)^n, \ldots \qquad \frac{1/9}{-1/3} = -\frac{1}{3}$$

Begin with $n = 1$.

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Write the first four terms of the geometric sequence whose $n$th term is $6(-2)^n$. Then find the common ratio of the consecutive terms.

In Example 1, notice that each of the geometric sequences has an $n$th term that is of the form $ar^n$, where the common ratio of the sequence is $r$. A geometric sequence may be thought of as an exponential function whose domain is the set of natural numbers.

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### The *n*th Term of a Geometric Sequence

The *n*th term of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

where $r$ is the common ratio of consecutive terms of the sequence. So, every geometric sequence can be written in the form below.

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ \ldots,\ a_n,\ \ldots$$
$$\downarrow\quad \downarrow\quad \downarrow\quad \downarrow\quad \downarrow\qquad\qquad \downarrow$$
$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ a_1r^4,\ \ldots,\ a_1r^{n-1},\ \ldots$$

When you know the *n*th term of a geometric sequence, multiply by $r$ to find the $(n + 1)$th term. That is, $a_{n+1} = a_n r$.

## EXAMPLE 2 Writing the Terms of a Geometric Sequence

Write the first five terms of the geometric sequence whose first term is $a_1 = 3$ and whose common ratio is $r = 2$. Then graph the terms on a set of coordinate axes.

**Solution** Starting with 3, repeatedly multiply by 2 to obtain the terms below.

$a_1 = 3$ — 1st term

$a_2 = 3(2^1) = 6$ — 2nd term

$a_3 = 3(2^2) = 12$ — 3rd term

$a_4 = 3(2^3) = 24$ — 4th term

$a_5 = 3(2^4) = 48$ — 5th term

Figure 9.1 shows the graph of the first five terms of this geometric sequence.

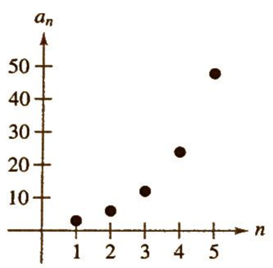

**Figure 9.1**

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Write the first five terms of the geometric sequence whose first term is $a_1 = 2$ and whose common ratio is $r = 4$. Then graph the terms on a set of coordinate axes.

## EXAMPLE 3 Finding a Term of a Geometric Sequence

Find the 15th term of the geometric sequence whose first term is 20 and whose common ratio is 1.05.

### Algebraic Solution

$a_n = a_1 r^{n-1}$ — Formula for *n*th term of a geometric sequence

$a_{15} = 20(1.05)^{15-1}$ — Substitute 20 for $a_1$, 1.05 for $r$, and 15 for $n$.

$\approx 39.60$ — Use a calculator.

### Numerical Solution

For this sequence, $r = 1.05$ and $a_1 = 20$. So, $a_n = 20(1.05)^{n-1}$. Use a graphing utility to create a table that shows the terms of the sequence.

| n | u(n) |
|---|---|
| 9 | 29.549 |
| 10 | 31.027 |
| 11 | 32.578 |
| 12 | 34.207 |
| 13 | 35.917 |
| 14 | 37.713 |
| 15 | 39.599 |

u(n)=39.59863199

The number in the 15th row is the 15th term of the sequence.

So, $a_{15} \approx 39.60$.

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Find the 12th term of the geometric sequence whose first term is 14 and whose common ratio is 1.2.

## EXAMPLE 4 Writing the *n*th Term of a Geometric Sequence

Find a formula for the *n*th term of the geometric sequence

$5, 15, 45, \ldots$

What is the 12th term of the sequence?

**Solution** The common ratio of this sequence is $r = 15/5 = 3$. The first term is $a_1 = 5$, so the formula for the *n*th term is

$$a_n = a_1 r^{n-1}$$
$$= 5(3)^{n-1}.$$

Use the formula for $a_n$ to find the 12th term of the sequence.

$$a_{12} = 5(3)^{12-1} \quad \text{Substitute 12 for } n.$$
$$= 5(177{,}147) \quad \text{Use a calculator.}$$
$$= 885{,}735. \quad \text{Multiply.}$$

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Find a formula for the *n*th term of the geometric sequence

$4, 20, 100, \ldots$

What is the 12th term of the sequence?

When you know *any* two terms of a geometric sequence, you can use that information to find *any other* term of the sequence.

**ALGEBRA HELP**

Remember that $r$ is the common ratio of consecutive terms of a geometric sequence. So, in Example 5

$$a_{10} = a_1 r^9$$
$$= a_1 \cdot r \cdot r \cdot r \cdot r^6$$
$$= a_1 \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \frac{a_4}{a_3} \cdot r^6$$
$$= a_4 r^6.$$

## EXAMPLE 5 Finding a Term of a Geometric Sequence

The 4th term of a geometric sequence is 125, and the 10th term is 125/64. Find the 14th term. (Assume that the terms of the sequence are positive.)

**Solution** The 10th term is related to the 4th term by the equation

$$a_{10} = a_4 r^6. \quad \text{Multiply fourth term by } r^{10-4}.$$

Use $a_{10} = 125/64$ and $a_4 = 125$ to solve for $r$.

$$\frac{125}{64} = 125r^6 \quad \text{Substitute } \tfrac{125}{64} \text{ for } a_{10} \text{ and 125 for } a_4.$$
$$\frac{1}{64} = r^6 \quad \text{Divide each side by 125.}$$
$$\frac{1}{2} = r \quad \text{Take the sixth root of each side.}$$

Multiply the 10th term by $r^{14-10} = r^4$ to obtain the 14th term.

$$a_{14} = a_{10}r^4 = \frac{125}{64}\left(\frac{1}{2}\right)^4 = \frac{125}{64}\left(\frac{1}{16}\right) = \frac{125}{1024}$$

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The second term of a geometric sequence is 6, and the fifth term is 81/4. Find the eighth term. (Assume that the terms of the sequence are positive.)

## The Sum of a Finite Geometric Sequence

The formula for the sum of a *finite* geometric sequence is as follows.

**The Sum of a Finite Geometric Sequence**

The sum of the finite geometric sequence

$$a_1, a_1r, a_1r^2, a_1r^3, a_1r^4, \ldots, a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by $S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right)$.

For a proof of this formula for the sum of a finite geometric sequence, see Proofs in Mathematics on page 687.

### EXAMPLE 6 Sum of a Finite Geometric Sequence

Find the sum $\sum_{i=1}^{12} 4(0.3)^{i-1}$.

**Solution** You have

$$\sum_{i=1}^{12} 4(0.3)^{i-1} = 4(0.3)^0 + 4(0.3)^1 + 4(0.3)^2 + \cdots + 4(0.3)^{11}.$$

Using $a_1 = 4$, $r = 0.3$, and $n = 12$, apply the formula for the sum of a finite geometric sequence.

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right) \qquad \text{Sum of a finite geometric sequence}$$

$$\sum_{i=1}^{12} 4(0.3)^{i-1} = 4\left[\frac{1-(0.3)^{12}}{1-0.3}\right] \qquad \text{Substitute 4 for } a_1\text{, 0.3 for } r\text{, and 12 for } n.$$

$$\approx 5.714 \qquad \text{Use a calculator.}$$

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Find the sum $\sum_{i=1}^{10} 2(0.25)^{i-1}$.

When using the formula for the sum of a finite geometric sequence, make sure that the sum is of the form

$$\sum_{i=1}^{n} a_1 r^{i-1}. \qquad \text{Exponent for } r \text{ is } i-1.$$

For a sum that is not of this form, you must rewrite the sum before applying the formula. For example, the sum $\sum_{i=1}^{12} 4(0.3)^i$ is evaluated as follows.

$$\sum_{i=1}^{12} 4(0.3)^i = \sum_{i=1}^{12} 4[(0.3)(0.3)^{i-1}] \qquad \text{Property of exponents}$$

$$= \sum_{i=1}^{12} 4(0.3)(0.3)^{i-1} \qquad \text{Associative Property}$$

$$= 4(0.3)\left[\frac{1-(0.3)^{12}}{1-0.3}\right] \qquad a_1 = 4(0.3),\ r = 0.3,\ n = 12$$

$$\approx 1.714$$

## Geometric Series

The sum of the terms of an infinite geometric *sequence* is called an **infinite geometric series** or simply a **geometric series.**

The formula for the sum of a *finite geometric sequence* can, depending on the value of $r$, be extended to produce a formula for the sum of an *infinite geometric series.* Specifically, if the common ratio $r$ has the property that $|r| < 1$, then it can be shown that $r^n$ approaches zero as $n$ increases without bound. Consequently,

$$a_1\left(\frac{1-r^n}{1-r}\right) \to a_1\left(\frac{1-0}{1-r}\right) \quad \text{as} \quad n \to \infty.$$

The following summarizes this result.

**The Sum of an Infinite Geometric Series**

If $|r| < 1$, then the infinite geometric series

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \cdot \cdot \cdot + a_1r^{n-1} + \cdot \cdot \cdot$$

has the sum

$$S = \sum_{i=0}^{\infty} a_1r^i = \frac{a_1}{1-r}.$$

Note that when $|r| \geq 1$, the series does not have a sum.

**EXAMPLE 7** **Finding the Sum of an Infinite Geometric Series**

Find each sum.

**a.** $\sum_{n=0}^{\infty} 4(0.6)^n$

**b.** $3 + 0.3 + 0.03 + 0.003 + \cdot \cdot \cdot$

**Solution**

**a.** $\sum_{n=0}^{\infty} 4(0.6)^n = 4 + 4(0.6) + 4(0.6)^2 + 4(0.6)^3 + \cdot \cdot \cdot + 4(0.6)^n + \cdot \cdot \cdot$

$$= \frac{4}{1-0.6} \qquad \frac{a_1}{1-r}$$

$$= 10$$

**b.** $3 + 0.3 + 0.03 + 0.003 + \cdot \cdot \cdot = 3 + 3(0.1) + 3(0.1)^2 + 3(0.1)^3 + \cdot \cdot \cdot$

$$= \frac{3}{1-0.1} \qquad \frac{a_1}{1-r}$$

$$= \frac{10}{3}$$

$$\approx 3.33$$

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Find each sum.

**a.** $\sum_{n=0}^{\infty} 5(0.5)^n$

**b.** $5 + 1 + 0.2 + 0.04 + \cdot \cdot \cdot$

## Application

**EXAMPLE 8** **Increasing Annuity**

*See LarsonPrecalculus.com for an interactive version of this type of example.*

An investor deposits \$50 on the first day of each month in an account that pays 3% interest, compounded monthly. What is the balance at the end of 2 years? (This type of investment plan is called an **increasing annuity.**)

**Solution** To find the balance in the account after 24 months, consider each of the 24 deposits separately. The first deposit will gain interest for 24 months, and its balance will be

$$A_{24} = 50\left(1 + \frac{0.03}{12}\right)^{24}$$

$$= 50(1.0025)^{24}.$$

The second deposit will gain interest for 23 months, and its balance will be

$$A_{23} = 50\left(1 + \frac{0.03}{12}\right)^{23}$$

$$= 50(1.0025)^{23}.$$

The last deposit will gain interest for only 1 month, and its balance will be

$$A_1 = 50\left(1 + \frac{0.03}{12}\right)^{1}$$

$$= 50(1.0025).$$

The total balance in the annuity will be the sum of the balances of the 24 deposits. Using the formula for the sum of a finite geometric sequence, with $A_1 = 50(1.0025)$, $r = 1.0025$, and $n = 24$, you have

$$S_n = A_1\left(\frac{1 - r^n}{1 - r}\right) \quad \text{Sum of a finite geometric sequence}$$

$$S_{24} = 50(1.0025)\left[\frac{1 - (1.0025)^{24}}{1 - 1.0025}\right] \quad \text{Substitute } 50(1.0025) \text{ for } A_1, 1.0025 \text{ for } r, \text{ and } 24 \text{ for } n.$$

$$\approx \$1238.23. \quad \text{Use a calculator.}$$

**REMARK** Recall from Section 3.1 that the formula for compound interest (for $n$ compoundings per year) is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

So, in Example 8, \$50 is the principal $P$, 0.03 is the annual interest rate $r$, 12 is the number $n$ of compoundings per year, and 2 is the time $t$ in years. When you substitute these values into the formula, you obtain

$$A = 50\left(1 + \frac{0.03}{12}\right)^{12(2)}$$

$$= 50\left(1 + \frac{0.03}{12}\right)^{24}.$$

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An investor deposits \$70 on the first day of each month in an account that pays 2% interest, compounded monthly. What is the balance at the end of 4 years?

### Summarize (Section 9.3)

1. State the definition of a geometric sequence *(page 629)* and state the formula for the $n$th term of a geometric sequence *(page 630)*. For examples of recognizing, writing, and finding the $n$th terms of geometric sequences, see Examples 1–5.
2. State the formula for the sum of a finite geometric sequence *(page 632)*. For an example of finding the sum of a finite geometric sequence, see Example 6.
3. State the formula for the sum of an infinite geometric series *(page 633)*. For an example of finding the sums of infinite geometric series, see Example 7.
4. Describe an example of how to use a geometric sequence to model and solve a real-life problem *(page 634, Example 8)*.