12.3 The Tangent Line Problem

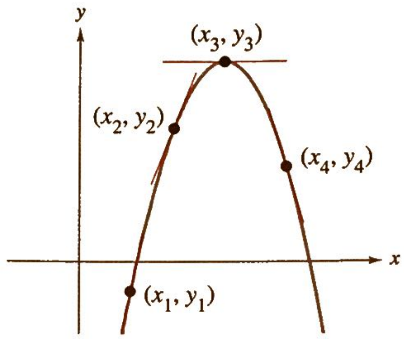
The slope of a tangent line to the graph of a function is useful for analyzing rates of change in real-life situations. For example, in Exercise 71 on page 847, you will use the slope of a tangent line to a graph to analyze the path of a ball.

- **Understand the tangent line problem.**
- **Use a tangent line to approximate the slope of a graph at a point.**
- **Use the limit definition of slope to find exact slopes of graphs.**
- **Find derivatives of functions and use derivatives to find slopes of graphs.**

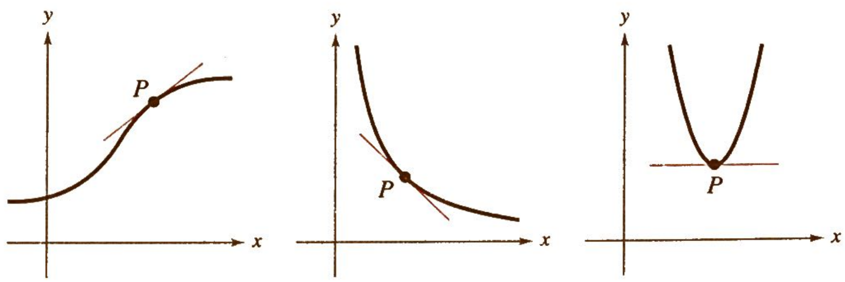
Tangent Line to a Graph

Calculus is a branch of mathematics that studies rates of change of functions. When you take a course in calculus, you will learn that rates of change have many applications.

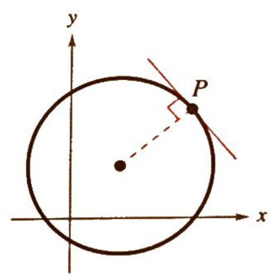
Earlier in the text, you learned that the slope of a line is a measure of the rate at which a line rises or falls. On a line, this rate (or slope) is the same at every point. For graphs other than lines, the rate at which the graph rises or falls changes from point to point. For example, the parabola shown below rises more quickly at the point (x_1, y_1) than at the point (x_2, y_2). At the vertex (x_3, y_3), the graph does not rise or fall (the slope is 0), and at the point (x_4, y_4), the graph falls.



To determine the rate at which a graph rises or falls at a *single point*, you can find the slope of the tangent line at that point. The **tangent line** to the graph of a function f at a point $P(x_1, y_1)$ is the line whose slope best approximates the slope of the graph at the point. The graphs below show other examples of tangent lines.



From geometry, you know that a line is tangent to a circle when the line intersects the circle at only one point, as shown at the right. Tangent lines to noncircular graphs, however, can intersect the graph at more than one point. For example, in the leftmost graph above, if the tangent line were extended, it would intersect the graph at a point other than the point of tangency.



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Slope of a Graph

A tangent line approximates the slope of the graph at a point, so the problem of finding the slope of a graph at a point is the same as finding the slope of the tangent line at the point.

EXAMPLE 1 Visually Approximating the Slope of a Graph

Use Figure 12.13 to approximate the slope of the graph of $f(x) = x^2$ at the point $(1, 1)$.

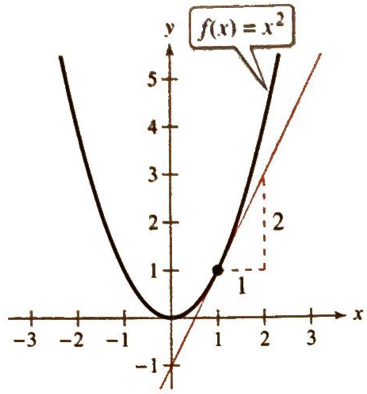

Figure 12.13

Solution From the graph of $f(x) = x^2$, notice that the tangent line at $(1, 1)$ rises approximately two units for each unit change in x. So, you can estimate the slope of the tangent line at $(1, 1)$ to be

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} \approx \frac{2}{1} = 2.$$

The tangent line at the point $(1, 1)$ has a slope of about 2, so the graph of f has a slope of about 2 at the point $(1, 1)$.

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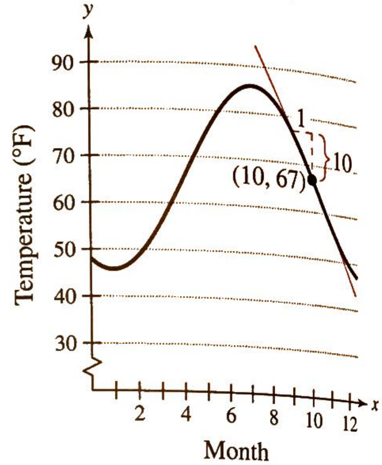
Use Figure 12.13 to approximate the slope of the graph of $f(x) = x^2$ at the point $(2, 4)$.

When you visually approximate the slope of a graph, remember that the scales on the horizontal and vertical axes may differ. When this happens (as it frequently does in applications), the slope of the tangent line is distorted, and you must account for the difference in scales.

EXAMPLE 2 Visually Approximating the Slope of a Graph

According to the National Climatic Data Center, the normal temperature in Dallas, Texas, for the month of October is about 67°F.

The figure at the right graphically depicts the monthly normal temperatures (in degrees Fahrenheit) for Dallas, Texas. Approximate the slope of this graph at the point shown and give a physical interpretation of the result. *(Source: National Climatic Data Center)*



Solution From the graph, the tangent line at the given point falls approximately 10 units for each one-unit change in x. So, you can estimate the slope at the given point to be

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} \approx \frac{-10}{1} = -10 \text{ degrees per month.}$$

This means that the monthly normal temperature in November is about 10 degrees lower than the monthly normal temperature in October.

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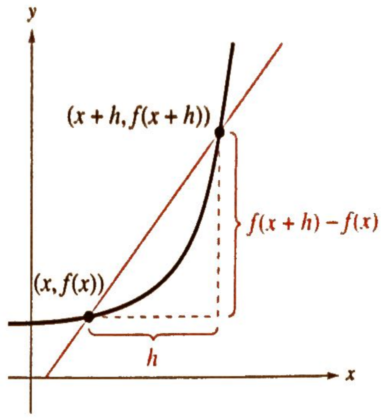
In Example 2, approximate the slope of the graph at the point $(4, 66)$ and give a physical interpretation of the result.

Slope and the Limit Process

In Examples 1 and 2, you approximated the slope of a graph at a point by creating a graph and then "eyeballing" the tangent line at the point of tangency. A more precise method of approximating tangent lines uses a **secant line** through the point of tangency and a second point on the graph, as shown at the right. If



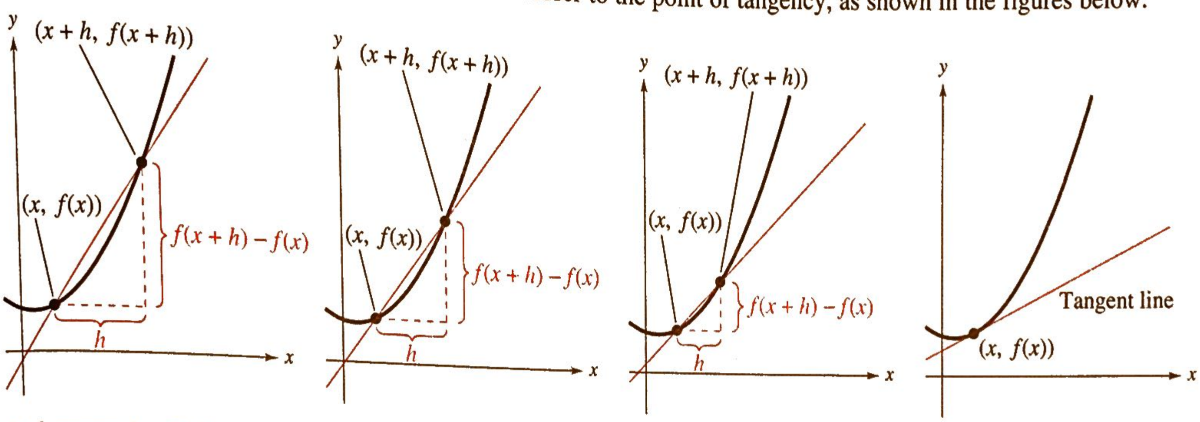
$$(x, f(x)) \quad \text{and} \quad (x + h, f(x + h))$$

are two points on the graph of f, and $(x, f(x))$ is the point of tangency, then the slope of the secant line through the two points is

$$m_{\text{sec}} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x + h) - f(x)}{h}.$$

Slope of secant line

Notice that the right side of this equation is a difference quotient. The denominator h is the *change in x*, and the numerator is the *change in y*. Using this method, you obtain more and more accurate approximations of the slope of the tangent line by choosing points closer and closer to the point of tangency, as shown in the figures below.



As h approaches 0, the secant line approaches the tangent line.

Using the limit process, you can find the *exact* slope of the tangent line at $(x, f(x))$.

> **Definition of the Slope of a Graph**
>
> The **slope** m of the graph of f at the point $(x, f(x))$ is equal to the slope of its tangent line at $(x, f(x))$, and is given by
>
> $$m = \lim_{h \to 0} m_{\text{sec}} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
>
> provided this limit exists.

As suggested in Section 12.2, the difference quotient is used frequently in calculus. Using the difference quotient to find the slope of a tangent line to a graph is a major concept of calculus.

EXAMPLE 3 Finding the Slope of a Graph

Find the slope of the graph of $f(x) = x^2$ at the point $(-2, 4)$.

Solution Find an expression that represents the slope of a secant line at $(-2, 4)$.

$$m_{sec} = \frac{f(-2+h) - f(-2)}{h} \quad \text{Set up difference quotient.}$$

$$= \frac{(-2+h)^2 - (-2)^2}{h} \quad \text{Substitute into } f(x) = x^2.$$

$$= \frac{4 - 4h + h^2 - 4}{h} \quad \text{Expand terms.}$$

$$= \frac{-4h + h^2}{h} \quad \text{Simplify.}$$

$$= \frac{h(-4+h)}{h} \quad \text{Factor and divide out.}$$

$$= -4 + h, \quad h \neq 0 \quad \text{Simplify.}$$

Next, find the limit of m_{sec} as h approaches 0.

$$m = \lim_{h \to 0} m_{sec}$$

$$= \lim_{h \to 0} (-4 + h)$$

$$= -4$$

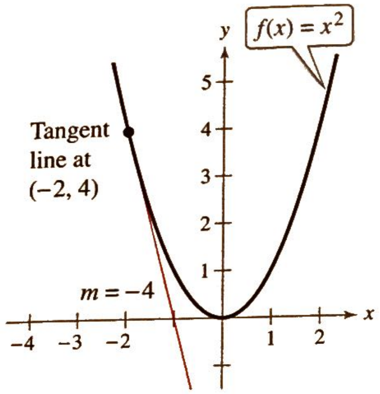


Figure 12.14

The graph has a slope of -4 at the point $(-2, 4)$, as shown in Figure 12.14.

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Use the limit process to find the slope of the graph of $g(x) = x^2 - 2x$ at the point $(3, 3)$.

EXAMPLE 4 Finding the Slope of a Graph

Find the slope of $f(x) = -2x + 4$.

Solution

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{Set up difference quotient.}$$

$$= \lim_{h \to 0} \frac{[-2(x+h) + 4] - (-2x + 4)}{h} \quad \text{Substitute into } f(x) = -2x + 4.$$

$$= \lim_{h \to 0} \frac{-2x - 2h + 4 + 2x - 4}{h} \quad \text{Expand terms.}$$

$$= \lim_{h \to 0} \frac{-2h}{h} \quad \text{Divide out.}$$

$$= -2 \quad \text{Simplify.}$$

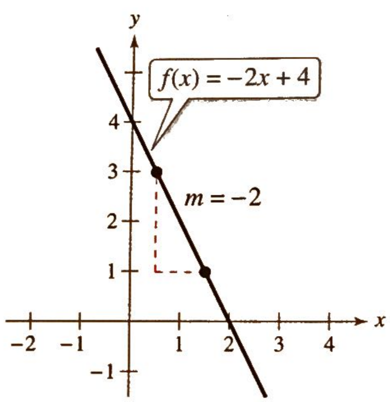

Figure 12.15

You know from your study of linear functions that the line $f(x) = -2x + 4$ has a slope of -2, as shown in Figure 12.15. This conclusion is consistent with that obtained by the limit definition of slope, as shown above.

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Use the limit process to find the slope of $f(x) = -3x + 4$.

It is important that you see the difference between the ways the difference quotients were set up in Examples 3 and 4. In Example 3, you found the slope of a graph at a specific point $(c, f(c))$. To find the slope in such a case, use the form of the difference quotient below.

$$m = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \qquad \text{Slope at specific point}$$

In Example 4, however, you found a *formula* for the slope at *any* point on the graph. In such cases, you should use x, rather than c, in the difference quotient.

$$m = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \qquad \text{Formula for slope}$$

This form will always produce a function of x, which can then be evaluated to find the slope at any desired point on the graph.

TECHNOLOGY Use the slopes found in Example 5 and the point-slope form of the equation of a line to find that the equations of the tangent lines at $(-1, 2)$ and $(2, 5)$ are $y = -2x$ and $y = 4x - 3$, respectively. Then use a graphing utility to graph the function and the tangent lines

$y_1 = x^2 + 1$

$y_2 = -2x$

$y_3 = 4x - 3$

in the same viewing window. You can verify the equations of the tangent lines using the *tangent* feature of the graphing utility.

EXAMPLE 5 Finding a Formula for the Slope of a Graph

Find a formula for the slope of the graph of $f(x) = x^2 + 1$. What are the slopes at the points $(-1, 2)$ and $(2, 5)$?

Solution

$$m_{sec} = \frac{f(x + h) - f(x)}{h} \qquad \text{Set up difference quotient.}$$

$$= \frac{[(x + h)^2 + 1] - (x^2 + 1)}{h} \qquad \text{Substitute into } f(x) = x^2 + 1.$$

$$= \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} \qquad \text{Expand terms.}$$

$$= \frac{2xh + h^2}{h} \qquad \text{Simplify.}$$

$$= \frac{\not{h}(2x + h)}{\not{h}} \qquad \text{Factor and divide out.}$$

$$= 2x + h, \quad h \neq 0 \qquad \text{Simplify.}$$

Next, find the limit of m_{sec} as h approaches 0.

$$m = \lim_{h \to 0} m_{sec} \qquad \text{Formula for slope}$$

$$= \lim_{h \to 0} (2x + h)$$

$$= 2x$$

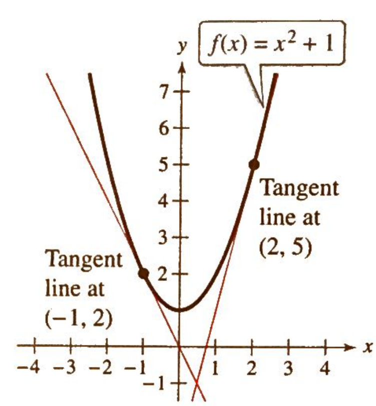
Using the formula $m = 2x$ for the slope at $(x, f(x))$, find the slope at the specified points. At $(-1, 2)$, the slope is

$$m = 2(-1) = -2$$

and at $(2, 5)$, the slope is

$$m = 2(2) = 4.$$

The graph of f is at the right.



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Find a formula for the slope of the graph of $f(x) = 2x^2 - 3$. What is the slope at the points $(-3, 15)$ and $(2, 5)$?

The Derivative of a Function

In Example 5, you started with the function $f(x) = x^2 + 1$ and used the limit process to derive another function, $m = 2x$, that represents the slope of the graph of f at the point $(x, f(x))$. This derived function is called the **derivative** of f at x. It is denoted by $f'(x)$, which is read as "f prime of x."

REMARK In Section 1.5, you studied the slope of a line, which represents the *average rate of change* over an interval. The derivative of a function is a formula that represents the *instantaneous rate of change* at a point.

Definition of a Derivative

The **derivative** of f at x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

Remember that the derivative $f'(x)$ is a formula for the slope of the tangent line to the graph of f at the point $(x, f(x))$.

EXAMPLE 6 **Finding a Derivative**

Find the derivative of

$$f(x) = 3x^2 - 2x.$$

Solution

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[3(x+h)^2 - 2(x+h)] - (3x^2 - 2x)}{h} \\
&= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 + 2x}{h} \\
&= \lim_{h \to 0} \frac{6xh + 3h^2 - 2h}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(6x + 3h - 2)}{\cancel{h}} \\
&= \lim_{h \to 0} (6x + 3h - 2) \\
&= 6x + 3(0) - 2 \\
&= 6x - 2
\end{aligned}$$

So, the derivative of $f(x) = 3x^2 - 2x$ is

$$f'(x) = 6x - 2.$$

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Find the derivative of

$$f(x) = x^3 + 2x.$$

Note that in addition to $f'(x)$, other notations can be used to denote the derivative of $y = f(x)$. The most common are

$$\frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad \text{and} \quad D_x[y].$$

EXAMPLE 7 Using the Derivative

See LarsonPrecalculus.com for an interactive version of this type of example.

Find $f'(x)$ for $f(x) = \sqrt{x}$. Then find the slopes of the graph of f at the points $(1, 1)$ and $(4, 2)$.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Direct substitution yields the indeterminate form $\frac{0}{0}$, so use the rationalizing technique discussed in Section 12.2 to find the limit.

$$f'(x) = \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

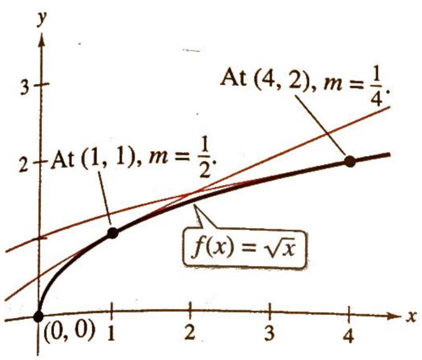
At the point $(1, 1)$, the slope is

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

At the point $(4, 2)$, the slope is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Figure 12.16 shows the graph of f.



The slope of f at $(x, f(x))$, $x > 0$, is $m = 1/(2\sqrt{x})$.

Figure 12.16

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Find $f'(x)$ for $f(x) = \sqrt{x - 1}$. Then find the slopes of the graph of f at the points $(2, 1)$ and $(10, 3)$.

Summarize (Section 12.3)

1. Explain what is meant by a tangent line and how to use a tangent line to approximate the slope of a graph at a point *(pages 839 and 840)*. For examples of approximating slopes of graphs at points, see Examples 1 and 2.
2. Explain how to use the limit definition of slope to find the exact slope of a graph *(page 841)*. For examples of using the limit definition of slope to find exact slopes of graphs, see Examples 3–5.
3. State the definition of the derivative of a function *(page 844)*. For examples of finding derivatives of functions and using derivatives to find slopes of graphs, see Examples 6 and 7.