12.4 Limits at Infinity and Limits of Sequences

Finding limits at infinity is useful in many types of real-life applications. For example, in Exercise 39 on page 856, you will use a limit at infinity to analyze the oxygen level in a pond.

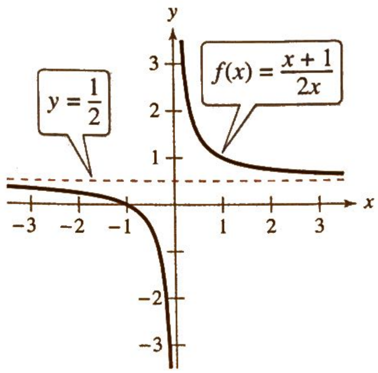
- Evaluate limits of functions at infinity.
- Find limits of sequences.

Limits at Infinity and Horizontal Asymptotes

As pointed out at the beginning of this chapter, there are two basic problems in calculus: finding tangent lines and finding the area of a region. In Section 12.3, you saw how limits can help you solve the tangent line problem. In this section and the next, you will see how a different type of limit, a *limit at infinity*, can help you solve the area problem. To get an idea of what is meant by a limit at infinity, consider the function

$$f(x) = \frac{x+1}{2x}.$$

The graph of f is shown below.



ALGEBRA HELP The function

$$f(x) = \frac{x+1}{2x}$$

is a rational function. To review rational functions, see Section 2.6.

From Section 2.6, you know that

$$y = \tfrac{1}{2}$$

is a horizontal asymptote of the graph of this function. Using limit notation, this is represented by the statements below.

$$\lim_{x\to -\infty} f(x) = \tfrac{1}{2} \qquad \text{Horizontal asymptote to the left}$$

$$\lim_{x\to \infty} f(x) = \tfrac{1}{2} \qquad \text{Horizontal asymptote to the right}$$

These limits mean that the value of $f(x)$ gets arbitrarily close to $\frac{1}{2}$ as x decreases or increases without bound.

Definition of Limits at Infinity

If f is a function and L_1 and L_2 are real numbers, then the statements

$$\lim_{x\to -\infty} f(x) = L_1 \qquad \text{Limit as } x \text{ approaches } -\infty$$

and

$$\lim_{x\to \infty} f(x) = L_2 \qquad \text{Limit as } x \text{ approaches } \infty$$

denote the **limits at infinity.** The first statement is read "*the limit of $f(x)$ as x approaches $-\infty$ is L_1*," and the second is read "*the limit of $f(x)$ as x approaches ∞ is L_2*."

Many of the properties of limits listed in Section 12.1 apply to limits at infinity. You can use these properties with the limits at infinity below to find more complicated limits, as shown in Example 1.

Limits at Infinity

If r is a positive real number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0. \qquad \text{Limit toward the right}$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0. \qquad \text{Limit toward the left}$$

EXAMPLE 1 Evaluating a Limit at Infinity

Find the limit.

$$\lim_{x \to \infty} \left(4 - \frac{3}{x^2}\right)$$

Algebraic Solution

Use the properties of limits listed in Section 12.1.

$$\lim_{x \to \infty} \left(4 - \frac{3}{x^2}\right) = \lim_{x \to \infty} 4 - \lim_{x \to \infty} \frac{3}{x^2}$$

$$= \lim_{x \to \infty} 4 - 3\left(\lim_{x \to \infty} \frac{1}{x^2}\right)$$

$$= 4 - 3(0)$$

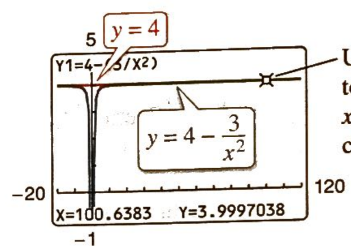
$$= 4$$

So, the limit of $f(x) = 4 - \frac{3}{x^2}$ as x approaches ∞ is 4.

Graphical Solution

Use a graphing utility to graph

$$y = 4 - \frac{3}{x^2}.$$



From the results shown above, estimate that the limit is 4. Also, note that the line $y = 4$ appears to be a horizontal asymptote to the right.

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Find the limit: $\lim_{x \to \infty} \frac{2}{x^2}$.

In the graphical solution to Example 1, it appears that the line $y = 4$ is also a horizontal asymptote *to the left*. Verify this by showing that

$$\lim_{x \to -\infty} \left(4 - \frac{3}{x^2}\right) = 4.$$

The graph of a rational function need not have a horizontal asymptote. When it does, however, its left and right horizontal asymptotes must be the same.

When evaluating limits at infinity for more complicated rational functions, divide the numerator and denominator by the highest power of the variable in the denominator. This enables you to evaluate each limit using the limits at infinity at the top of this page.

EXAMPLE 2 Evaluating Limits at Infinity

See LarsonPrecalculus.com for an interactive version of this type of example.

Find the limit as x approaches ∞ (if it exists) for each function.

a. $f(x) = \dfrac{-2x + 3}{3x^2 + 1}$

b. $f(x) = \dfrac{-2x^2 + 3}{3x^2 + 1}$

c. $f(x) = \dfrac{-2x^3 + 3}{3x^2 + 1}$

Solution In each case, begin by dividing both the numerator and denominator by x^2, the highest power of x in the denominator.

a. $$\lim_{x\to\infty} \frac{-2x + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{-\dfrac{2}{x} + \dfrac{3}{x^2}}{3 + \dfrac{1}{x^2}} = \frac{-0 + 0}{3 + 0} = 0$$

b. $$\lim_{x\to\infty} \frac{-2x^2 + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{-2 + \dfrac{3}{x^2}}{3 + \dfrac{1}{x^2}} = \frac{-2 + 0}{3 + 0} = -\frac{2}{3}$$

c. $$\lim_{x\to\infty} \frac{-2x^3 + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{-2x + \dfrac{3}{x^2}}{3 + \dfrac{1}{x^2}}$$

In this case, the limit does not exist because the numerator decreases without bound as the denominator approaches 3.

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Find the limit as x approaches ∞ (if it exists) for each function.

a. $f(x) = \dfrac{2x}{1 - x^2}$

b. $f(x) = \dfrac{2x}{1 - x}$

c. $f(x) = \dfrac{2x^2}{1 - x}$

In Example 2, observe that when the degree of the numerator is less than the degree of the denominator, as in part (a), the limit is 0. When the degrees of the numerator and denominator are equal, as in part (b), the limit is the ratio of the leading coefficients. When the degree of the numerator is greater than the degree of the denominator, as in part (c), the limit does not exist.

This result seems reasonable when you realize that for large values of x, the highest-powered term of a polynomial is the most "influential" term. That is, a polynomial tends to behave as its highest-powered term behaves as x approaches positive or negative infinity.

Limits at Infinity for Rational Functions

Consider the rational function

$$f(x) = \frac{N(x)}{D(x)}$$

where

$$N(x) = a_n x^n + \cdots + a_0 \quad \text{and} \quad D(x) = b_m x^m + \cdots + b_0.$$

The limit of $f(x)$ as x approaches positive or negative infinity is as follows.

$$\lim_{x \to \pm\infty} f(x) = \begin{cases} 0, & n < m \\ \dfrac{a_n}{b_m}, & n = m \end{cases}$$

If $n > m$, then the limit does not exist.

EXAMPLE 3 Finding the Average Cost

Mobile phone protective cases accounted for 20.5% of the global mobile phone accessories market in 2014. Industry analysts predict the global annual sales of mobile phone accessories to exceed \$120 billion by 2025. *(Source: Future Market Insights)*

You are manufacturing mobile phone protective cases that cost \$4.50 per case to produce. Your initial investment is \$20,000, which implies that the total cost C of producing x cases is given by $C = 4.50x + 20,000$. The average cost $\overline{C}$ per case is given by

$$\overline{C} = \frac{C}{x} = \frac{4.50x + 20,000}{x}.$$

Find the average cost per case when (a) $x = 1000$, (b) $x = 10,000$, and (c) $x = 100,000$. (d) What is the limit of $\overline{C}$ as x approaches infinity?

Solution

a. When $x = 1000$,

$$\overline{C} = \frac{4.50(1000) + 20,000}{1000}$$

$$= \$24.50.$$

b. When $x = 10,000$,

$$\overline{C} = \frac{4.50(10,000) + 20,000}{10,000}$$

$$= \$6.50.$$

c. When $x = 100,000$,

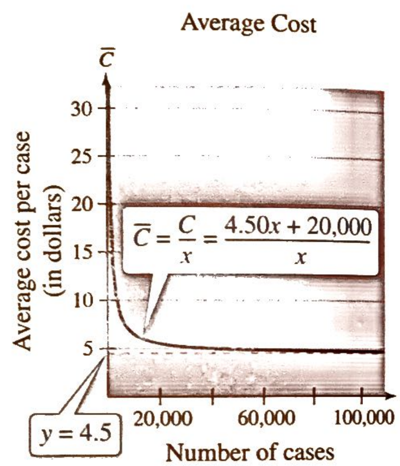
$$\overline{C} = \frac{4.50(100,000) + 20,000}{100,000}$$

$$= \$4.70.$$

d. As x approaches infinity, the limit of $\overline{C}$ is

$$\lim_{x \to \infty} \frac{4.50x + 20,000}{x} = \$4.50.$$

The graph of $\overline{C}$ is at the right.



As $x \to \infty$, the average cost per case approaches \$4.50.

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Repeat Example 3 for the same initial investment, but a cost per case of \$4.00.

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ALGEBRA HELP To review sequences, see Sections 9.1–9.3.

TECHNOLOGY There are a number of ways to use a graphing utility to generate the terms of a sequence. For example, to display the first 10 terms of the sequence

$$a_n = \frac{1}{2^n}$$

use the *sequence* feature or the *table* feature.

Limits of Sequences

Limits of sequences have many of the same properties as limits of functions. For example, consider the sequence whose nth term is $a_n = 1/2^n$.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

As n increases without bound, the terms of this sequence get closer and closer to 0, and the sequence is said to **converge** to 0. Using limit notation, this is represented as

$$\lim_{n\to\infty} (1/2^n) = 0.$$

So, the limit of a function can be used to evaluate the limit of a sequence.

Limit of a Sequence

Let f be a function of a real variable such that

$$\lim_{x\to\infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n\to\infty} a_n = L.$$

A sequence that does not converge is said to **diverge.** For example, the terms of the sequence $1, -1, 1, -1, 1, \ldots$ oscillate between 1 and -1. This sequence diverges because it does not converge to a unique number.

EXAMPLE 4 Finding the Limit of a Sequence

Write the first five terms of each sequence and find the limit of the sequence. (Assume that n begins with 1.)

a. $a_n = \dfrac{2n+1}{n+4}$ **b.** $a_n = \dfrac{2n+1}{n^2+4}$ **c.** $a_n = \dfrac{2n^2+1}{4n^2}$

Solution

a. The first five terms are $a_1 = \frac{3}{5}$, $a_2 = \frac{5}{6}$, $a_3 = 1$, $a_4 = \frac{9}{8}$, and $a_5 = \frac{11}{9}$. The limit of the sequence is

$$\lim_{n\to\infty} \frac{2n+1}{n+4} = 2.$$

b. The first five terms are $a_1 = \frac{3}{5}$, $a_2 = \frac{5}{8}$, $a_3 = \frac{7}{13}$, $a_4 = \frac{9}{20}$, and $a_5 = \frac{11}{29}$. The limit of the sequence is

$$\lim_{n\to\infty} \frac{2n+1}{n^2+4} = 0.$$

c. The first five terms are $a_1 = \frac{3}{4}$, $a_2 = \frac{9}{16}$, $a_3 = \frac{19}{36}$, $a_4 = \frac{33}{64}$, and $a_5 = \frac{51}{100}$. The limit of the sequence is

$$\lim_{n\to\infty} \frac{2n^2+1}{4n^2} = \frac{1}{2}.$$

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Write the first five terms of each sequence and find the limit of the sequence. (Assume that n begins with 1.)

a. $a_n = \dfrac{n+2}{2n-1}$ **b.** $a_n = \dfrac{n^3+2}{3n^3}$ **c.** $a_n = \dfrac{n+2}{3n^2}$

In the next section, you will encounter limits of sequences such as that shown in Example 5. A strategy for evaluating such limits is to begin by writing the nth term in standard rational function form. Then determine the limit by comparing the degrees of the numerator and denominator, as shown on page 852.

EXAMPLE 5 Finding the Limit of a Sequence

Find the limit of the sequence whose nth term is

$$a_n = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right].$$

Algebraic Solution

Begin by writing the nth term in standard rational function form—as the ratio of two polynomials.

$$a_n = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$ Write original nth term.

$$= \frac{8(n)(n+1)(2n+1)}{6n^3}$$ Multiply fractions.

$$= \frac{8n^3 + 12n^2 + 4n}{3n^3}$$ Write in standard rational form.

This form shows that the degree of the numerator is equal to the degree of the denominator. So, the limit of the sequence is the ratio of the leading coefficients.

$$\lim_{n\to\infty} \frac{8n^3 + 12n^2 + 4n}{3n^3} = \frac{8}{3}$$

Numerical Solution

Construct a table that shows the value of a_n as n becomes larger and larger.

n	a_n
1	8
10	3.08
100	2.7068
1000	2.6707
10,000	2.6671
100,000	2.6667
1,000,000	2.6667

Notice from the table that as n approaches ∞, a_n gets closer and closer to

$$2.6667 \approx \frac{8}{3}.$$

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Find the limit of each sequence whose nth term is given.

a. $a_n = \frac{5}{n^3}\left[\frac{(n+1)(n-1)(2n)}{7}\right]$

b. $a_n = \frac{2}{n}\left(n - \frac{2}{n}\left[\frac{n(n-1)}{4}\right]\right)$

Summarize (Section 12.4)

1. Define and explain how to evaluate limits of a function at infinity *(pages 849 and 850)*. For examples of evaluating limits at infinity, see Examples 1 and 2.
2. Explain how to evaluate limits at infinity for rational functions *(pages 851 and 852)*. For a real-life example of evaluating the limit at infinity for a rational function, see Example 3.
3. Explain how to find the limit of a sequence *(page 853)*. For examples of finding limits of sequences, see Examples 4 and 5.